

DEPARTMENT of MATHEMATICS

MAT302-120

Instructor:

CLT:

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OBJECTIVE: To calculate Riemann sums for a function and visualize the Riemann sums as an animated series of rectangles

> Typesetting:-RuleAssistant();

a) The integral value of the function $f(x)=x \sin(3 x) \, dx$ can be expressed as follows using maple;

> $f:=x \rightarrow x \cdot \sin(3 x);$

$$f:=x \mapsto x \sin(3 x) \quad (1)$$

> with(Student[Calculus1]):

> $\text{Int}\left(f(x), x=0 \ldots \frac{\pi}{2}\right);$

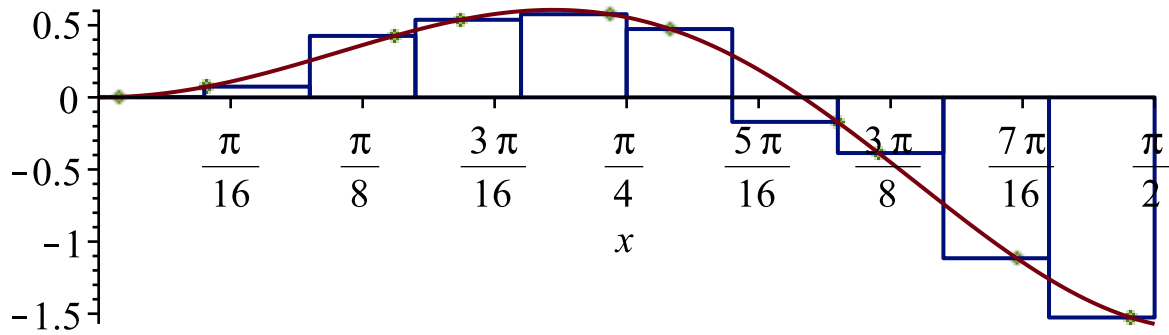
$$\int_0^{\frac{\pi}{2}} x \sin(3 x) \, dx \quad (2)$$

> evalf(%);

$$-0.1111111111 \quad (3)$$

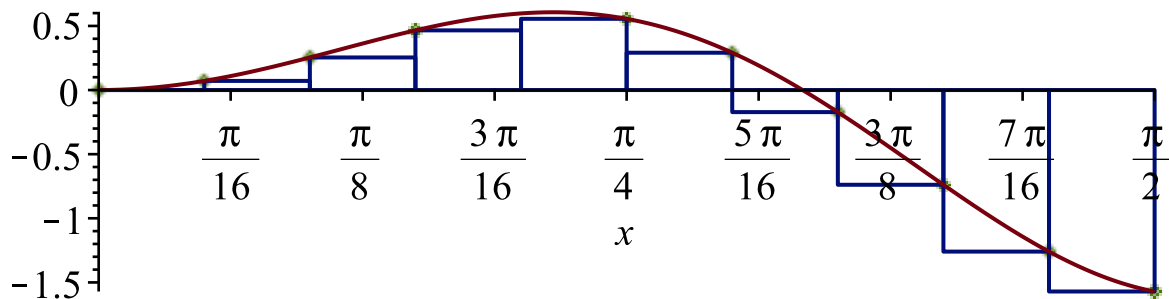
b) Using ApproximateInt command from the Student[Calculus1] package, we construct the rectangles that represent the random, the lower and the upper sums Riemann sum of n=10 subintervals as shown below.

> $\text{ApproximateInt}\left(f(x), x=0 \ldots \frac{\pi}{2}, \text{method}=\text{random}, \text{partition}=10, \text{output}=\text{plot}\right);$



An approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$ with randomly selected points, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.1738457885 . Number of subintervals used: 10.

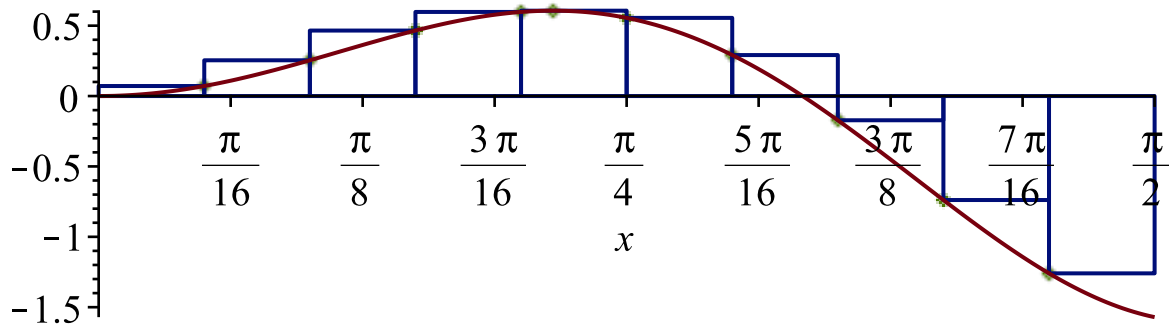
> `ApproximateInt` $\left(f(x), x=0..\frac{\pi}{2}, \text{method}=\text{lower}, \text{partition}=10, \text{output}=\text{plot}\right);$



A lower Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where

$f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.3304258828 . Number of subintervals used: 10.

> `ApproximateInt` $\left(f(x), x=0..\frac{\pi}{2}, \text{method}=\text{upper}, \text{partition}=10, \text{output}=\text{plot}\right);$



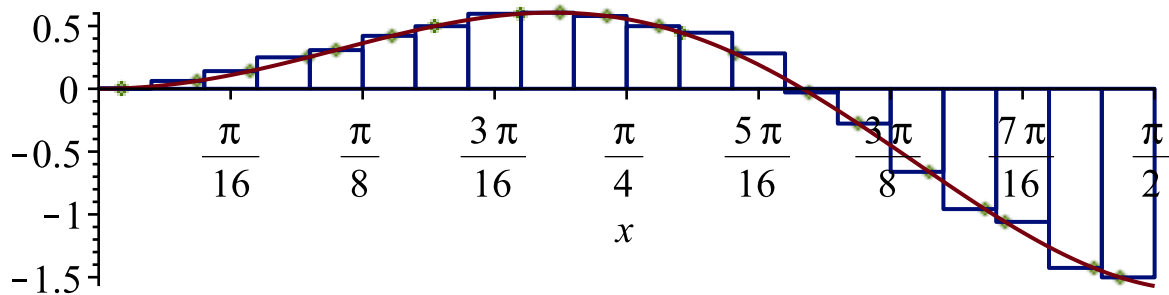
An upper Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where

$f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is 0.1054593127. Number of subintervals used: 10.

The random approximation gives a value between the lower and upper approximations where $n = 10$ subintervals which are still far from the "true" value -0.1111111111 of the definite integral.

c) Repeating the process b above for $n = 20$ subintervals and further concentrating on the lower and the upper Riemanns sum of $n = 40, 80, 160$, and 320 , we get the following result.

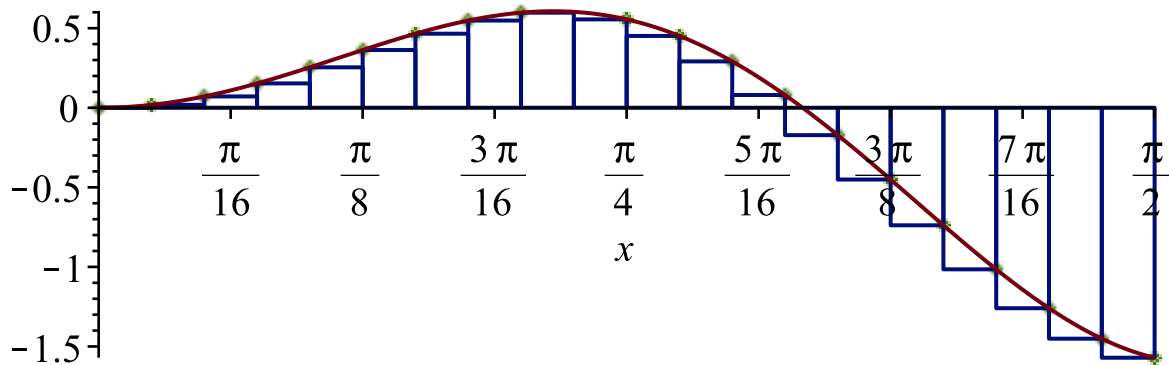
> `ApproximateInt` $\left(f(x), x = 0 \dots \frac{\pi}{2}, \text{method} = \text{random}, \text{partition} = 20, \text{output} = \text{plot}\right);$



An approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$ with randomly selected points, where

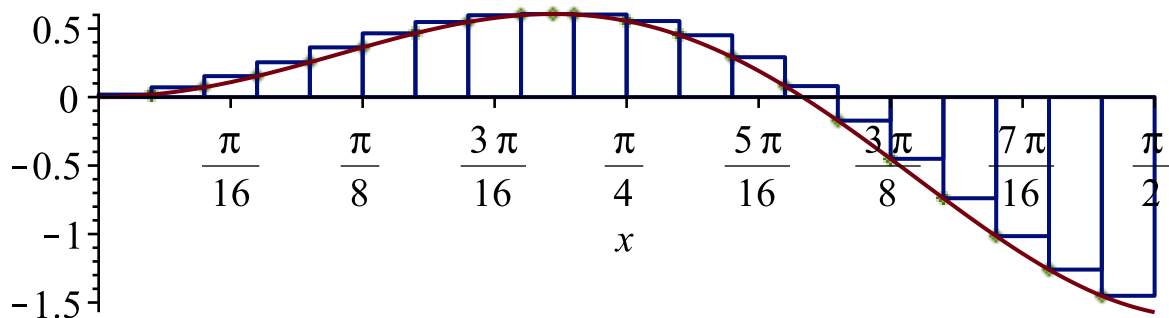
$f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.0955807057 . Number of subintervals used: 20.

> `ApproximateInt` $\left(f(x), x = 0 \dots \frac{\pi}{2}, \text{method} = \text{lower}, \text{partition} = 20, \text{output} = \text{plot}\right);$



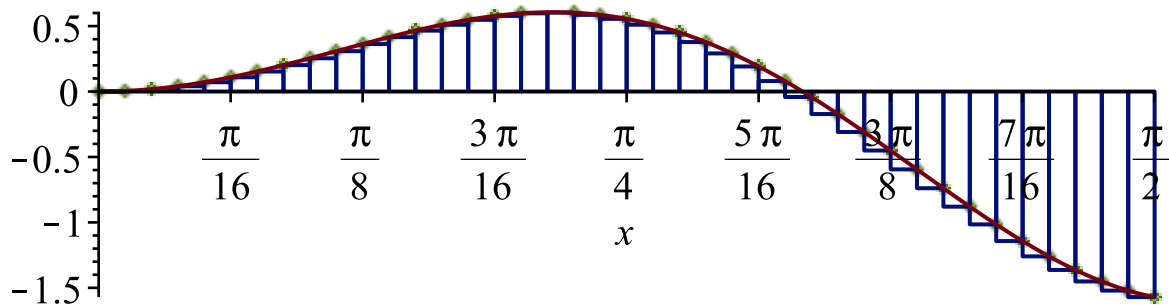
A lower Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.2206472298 . Number of subintervals used: 20.

> `ApproximateInt` $\left(f(x), x=0 \dots \frac{\pi}{2}, \text{method} = \text{upper}, \text{partition} = 20, \text{output} = \text{plot}\right);$



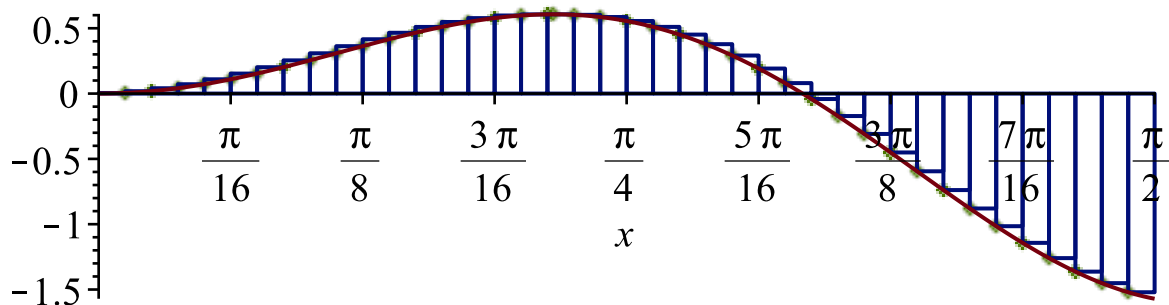
An upper Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.0023017710 . Number of subintervals used: 20.

> `ApproximateInt` $\left(f(x), x=0 \dots \frac{\pi}{2}, \text{method} = \text{lower}, \text{partition} = 40, \text{output} = \text{plot}\right);$



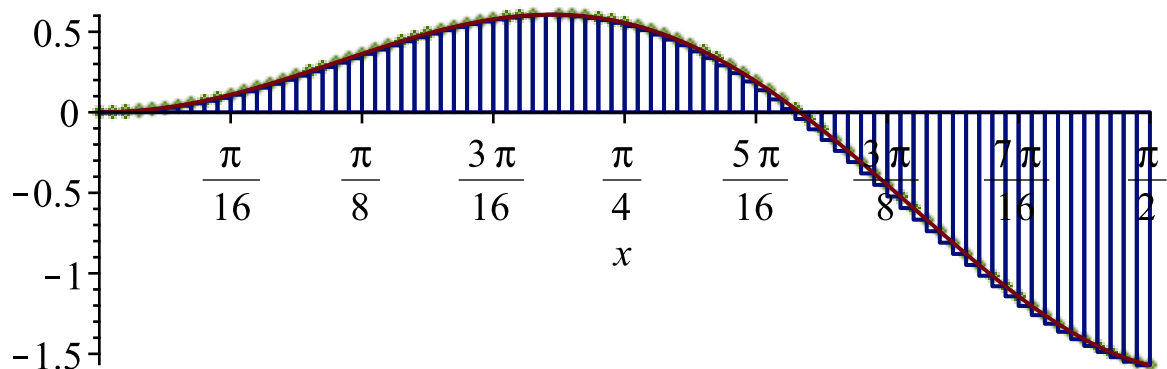
A lower Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.1658902306 . Number of subintervals used: 40.

> `ApproximateInt` $\left(f(x), x=0..\frac{\pi}{2}, \text{method}=\text{upper}, \text{partition}=40, \text{output}=\text{plot}\right);$



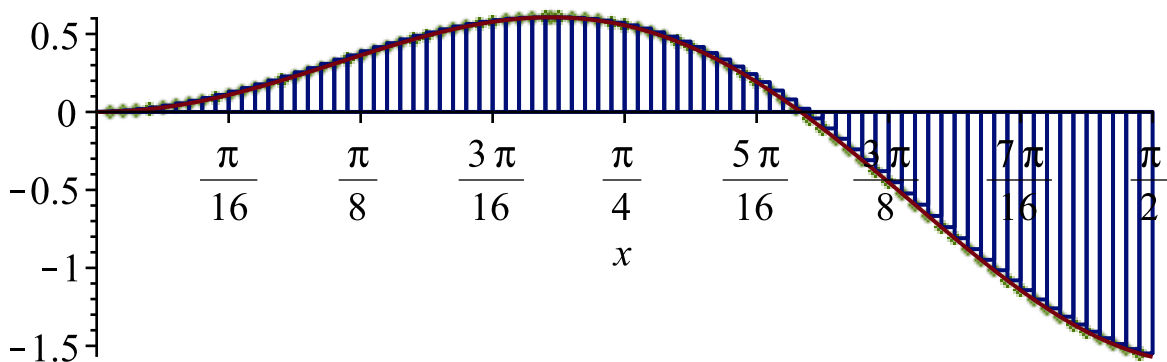
An upper Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.05657730508 . Number of subintervals used: 40.

> `ApproximateInt` $\left(f(x), x=0..\frac{\pi}{2}, \text{method}=\text{lower}, \text{partition}=80, \text{output}=\text{plot}\right);$



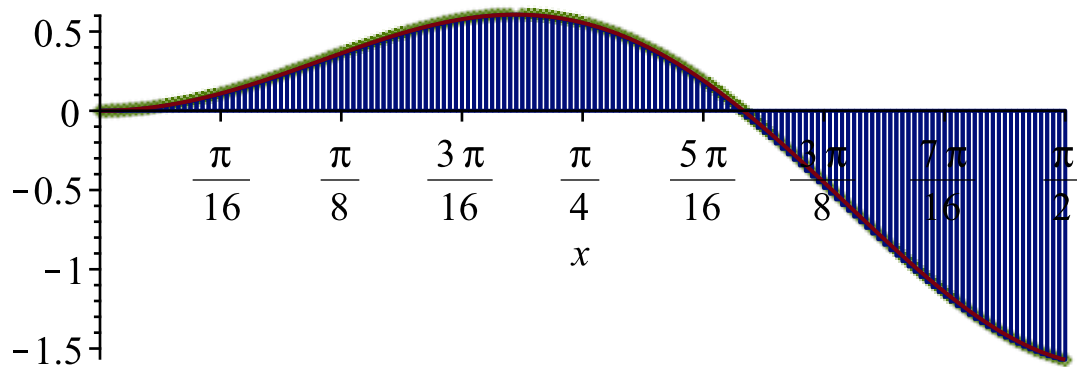
A lower Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.1384685042 . Number of subintervals used: 80.

> `ApproximateInt` $\left(f(x), x=0 \dots \frac{\pi}{2}, \text{method}=\text{upper}, \text{partition}=80, \text{output}=\text{plot}\right);$



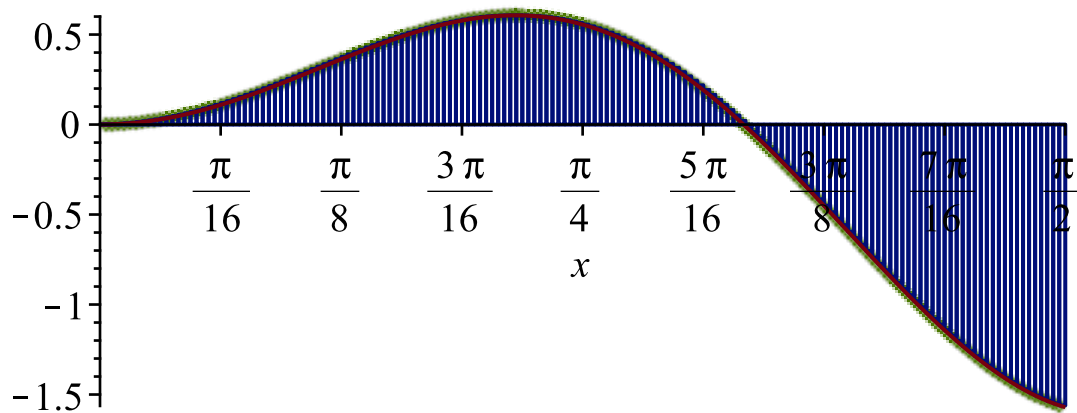
An upper Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.0838120414 . Number of subintervals used: 80.

> `ApproximateInt` $\left(f(x), x=0 \dots \frac{\pi}{2}, \text{method}=\text{lower}, \text{partition}=160, \text{output}=\text{plot}\right);$



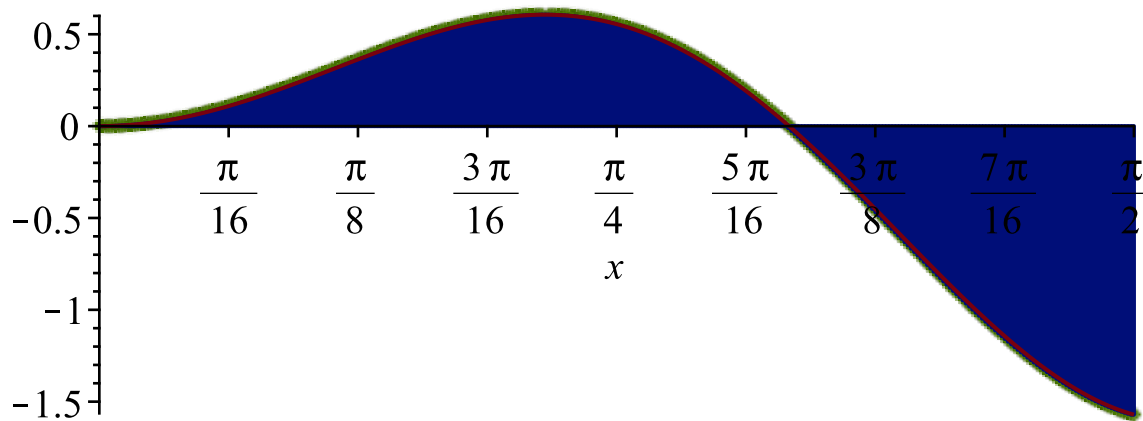
A lower Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.1247846924 . Number of subintervals used: 160.

> `ApproximateInt` $\left(f(x), x=0..\frac{\pi}{2}, \text{method}=\text{upper}, \text{partition}=160, \text{output}=\text{plot}\right);$



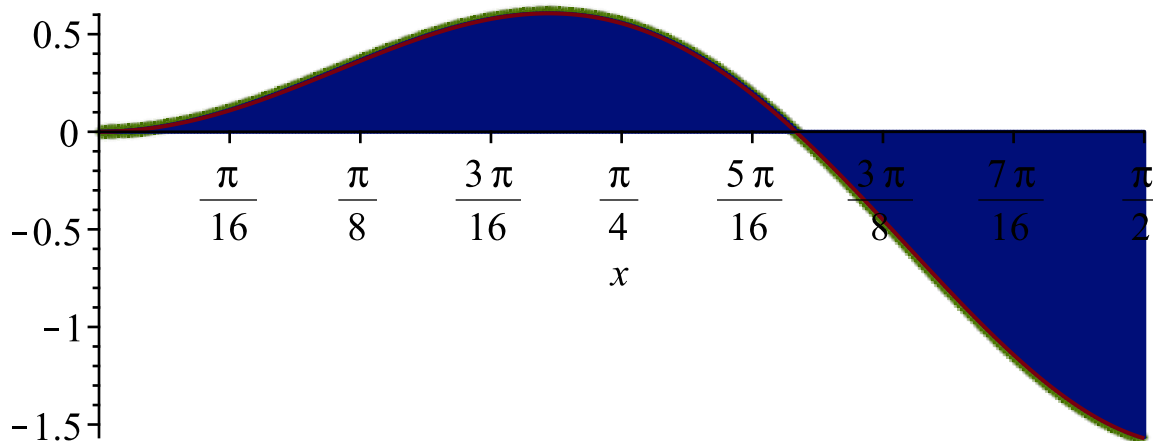
An upper Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.0974535425 . Number of subintervals used: 160.

> `ApproximateInt` $\left(f(x), x=0..\frac{\pi}{2}, \text{method}=\text{lower}, \text{partition}=320, \text{output}=\text{plot}\right);$



A lower Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.1179458937 . Number of subintervals used: 320.

> *ApproximateInt* $\left(f(x), x = 0 .. \frac{\pi}{2}, \text{method} = \text{upper}, \text{partition} = 320, \text{output} = \text{plot}\right);$



An upper Riemann sum approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$, where

$f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.1042803186 . Number of subintervals used: 320.

The partition became so dense as the number of subintervals increases that we can no longer distinguish individual subintervals. The approximations converge to 1 slowly and therefore the situation may look inconclusive.

d) When the number of subintervals are increased to $n=1000$, the following observation is made.

```
> ApproximateInt( $f(x), x = 0 \dots \frac{\pi}{2}, method = lower, partition = 1000, output = value$ );
```

-0.1132978095 (4)

```
> ApproximateInt( $f(x), x = 0 \dots \frac{\pi}{2}, method = upper, partition = 1000, output = value$ );
```

-0.1089248208 (5)

Even with such a dense partition we are still getting only two decimal digits right. Now we are convinced in convergence of Riemann sums to the integral/area, but feel a need to have better approximation tools. We will try the random, trapezoidal and Simpson methods,.

```
> ApproximateInt( $f(x), x = 0 \dots \frac{\pi}{2}, method = random, partition = 1000, output = value$ ) :
```

```
> evalf(%);
```

-0.1110919492 (6)

```
> ApproximateInt( $f(x), x = 0 \dots \frac{\pi}{2}, method = trapezoid, partition = 1000, output = value$ ) :
```

```
> evalf(%);
```

-0.1111113167 (7)

```
> ApproximateInt( $f(x), x = 0 \dots \frac{\pi}{2}, method = simpson, partition = 1000, output = value$ ) :
```

```
> evalf(%);
```

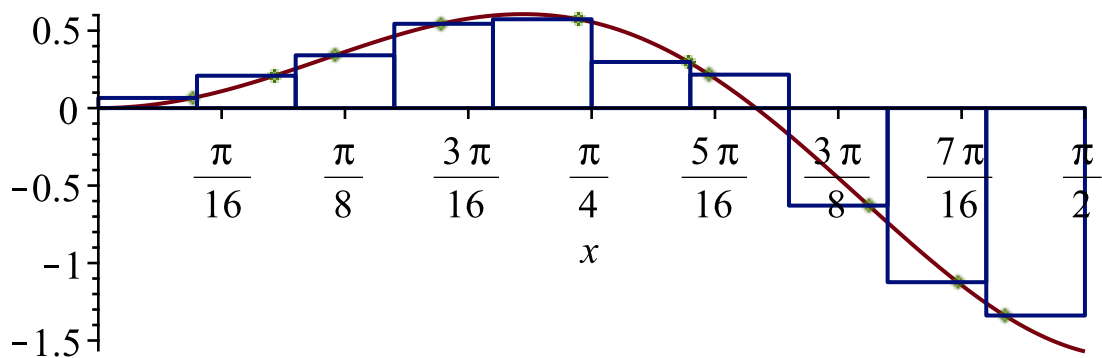
-0.1111111109 (8)

The random method wins over both lower sums and upper sums with three correct digits vs two, The trapezoidal method won over random with five correct digits, and Simpson method was even better and gave eight right digits under the same partition.

e) Finally, we run ApproximateInt in the animation mode to visualize the approximation of an area by Riemann sums.

```
> ApproximateInt( $f(x), x = 0 \dots \frac{\pi}{2}, method = random, partition = 10, output = animation, iterations$ 
```

= 7);



An animated approximation of $\int_0^{\frac{1}{2}\pi} f(x) \, dx$ with randomly selected points, where $f(x) = x \sin(3x)$ and the partition is uniform. The approximate value of the integral is -0.1326870558. Number of subintervals used: 10.

